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IDEAL THEORY IN NEAR-SEMIRINGS AND ITS APPLICATION TO AUTOMATA

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ABSTRACT. In this paper we develop ideal theory in near-semirings. We use the ideal theory to find the necessary and sufficient conditions for a linear sequential machine to be minimal.

1. INTRODUCTION

It has been shown that a homomorphic group-automaton $\mathcal{A} = (Q, A, B, F, G)$, where Q is a state set, A is an input set and B is an output set are groups and $F : Q \times A \to Q$ and $G : Q \times A \to B$, the state-transition function and output function respectively, are homomorphisms, is minimal if and only if the $N(\mathcal{A})$-group Q is generated by 0 and does not contain non-zero ideals which are annihilated by g_0 where $g_0 : Q \to B$ ([3], Theorem 9.259). Pilz [3] considered linear sequential machines in which the state set forms a group.

Krishna and Chatterjee [2] considered a generalized linear sequential machine $\mathcal{M} = (Q, A, B, F, G)$ where Q, A, B are semigroups and R-semimodules for some semiring R and $F : Q \times A \to Q$ and $G : Q \times A \to B$ are R-homomorphisms. They have obtained a necessary condition for the above generalized sequential machine to be minimal. So naturally one is interested to find a necessary and sufficient conditions for the above generalized linear sequential machine to be minimal. To achieve that, we develop ideal theory in a

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S-semigroup Γ, where S is a near-semiring. Using this ideal theory we find the necessary and sufficient conditions for a generalized linear sequential machine to be minimal. For the terminology and notation used in this paper we refer to Pilz [3], Krishna and Chatterjee [2].

2. NEAR-SEMIRINGS

A near- semiring is a nonempty set S with two binary operations '+'and '.' such that

(1) $(S,+)$ is a semigroup with identity 0,
(2) $(S,.)$ is a semigroup ,
(3) $(x+y)z = xz + yz$ for all $x,y,z \in S$, and
(4) $0s = 0$ for all $s \in S$.

In the near-semiring $(S,+,.)$, if $(S,.)$ has identity then S is a near-semiring with identity. Now we give a natural example of the near-semiring. Let $(\Gamma,+)$ be a semigroup with identity 0. If $M(\Gamma)$ is the set of all mappings from Γ into Γ then $M(\Gamma)$ is a near-semiring under pointwise addition and composition. $M(\Gamma)$ is neither a ring, nor a near-ring, nor a semiring. A semigroup $(S,+)$ is an inverse semigroup if for each $a \in S$, there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. Then a' is the additive inverse of a. A near-semiring $(S,+,.)$ is an additive inverse near-semiring if $(S,+)$ is an inverse semigroup. If A and B are any two non-empty sets of S, we define $AB = \{ab | a \in A, b \in B\}$. For $x,y \in S$, $x = (x')'$, $(x+y)' = y' + x'$ and $(xy)' = x'y$. We have $E^+(S) = \{a \in S : a + a = a\}$.

The properties of additive inverse semiring were obtained by Bandelt and Petrich [1] and the properties of regularity in an additive inverse semiring were obtained by Sen and Maity [4]. They have assumed the three conditions.

(1) $a(a+a') = (a+a')$
(2) $a(b+b') = (b+b')a$
(3) $a + a(b+b') = a$.

An element of $M(\Gamma)$ is said to be an affine mapping if it is a sum of an endomorphism and a constant map on Γ. The set of affine mappings on Γ is a subsemigroup of $M(\Gamma)$, denoted by $M_{aff}(\Gamma)$. Throughout this paper S denotes a near-semiring unless otherwise specified.

3. Ideal theory

Now we develop ideal theory in a S-semigroup Γ.

Definition 3.1. *Let S be a near-semiring. A semigroup $(\Gamma, +)$ is said to be an S-semigroup if there exists a mapping $(x, \gamma) \mapsto x\gamma$ of $S \times \Gamma \longrightarrow \Gamma$ such that for all $x, y \in S$, $\gamma \in \Gamma$,*

(1) $(x + y)\gamma = x\gamma + y\gamma$,
(2) $(xy)\gamma = x(y\gamma)$, *and*
(3) $0\gamma = 0_\Gamma$, *where 0_Γ is the zero of Γ.*

Definition 3.2. *A subsemigroup Δ of ${}_S\Gamma$ with $S\Delta \subseteq \Delta$ is said to be an S-subsemigroup of Γ.*

Definition 3.3. *Let ${}_S\Gamma_1$, ${}_S\Gamma_2$ be S-semigroups. A map $f : {}_S\Gamma_1 \to {}_S\Gamma_2$ is called an S-homomorphism if $f(\gamma + \gamma_1) = f(\gamma) + f(\gamma_1)$ and $f(s\gamma) = sf(\gamma)$ for all $\gamma, \gamma_1 \in {}_S\Gamma_1$ and $s \in S$.*

Note that $f(0_{\Gamma_1}) = 0_{\Gamma_2}$.

Definition 3.4. *If f is an S-homomorphism of Γ_1 into Γ_2, then the kernel of f is defined by $K = \{\gamma_1 \in \Gamma_1 | f(\gamma_1) = 0_{\Gamma_2}\}$.*

Hereafter $(\Gamma, +)$ is assumed to be inverse semigroup with $E^+(\Gamma)$ in the center of $(\Gamma, +)$.

Definition 3.5. *A non-empty subset I of an S-semigroup Γ is an ideal of ${}_S\Gamma$ ($I \trianglelefteq_S \Gamma$) if*

(1) $E^+(\Gamma) \subseteq I$,
(2) $i_1 + i_2' \in I$ *for all* $i_1, i_2 \in I$,
(3) $\gamma + i + \gamma' \in I$ *for all* $\gamma \in \Gamma$, $i \in I$,
(4) $s(i + \gamma) + (s\gamma)' \in I$ *for all* $\gamma \in \Gamma, i \in I$ *and* $s \in S$,
(5) *If* $e + \gamma \in I$ *implies* $\gamma \in I$ *for any* $e \in E^+(\Gamma)$.

Theorem 3.1. *If a non-empty subset I of an S-semigroup Γ satisfies the conditions $(1), (2), (3), (4)$ and (5) given above then I is the kernel of an S-homomorphism.*

Proof. Define the relation ρ on Γ by $a\rho b$ for all $a, b \in \Gamma$ if and only if $i_1 + a = i_2 + b$ for some $i_1, i_2 \in I$. Clearly ρ is reflexive and symmetric. Now we claim that ρ is transitive. Assume that $a\rho b$ and $b\rho c$. Then $i_1 + a = i_2 + b$ and $i_3 + b = i_4 + c$ for some $i_1, i_2, i_3, i_4 \in I$. Now $i_2+i_3+b = i_2+i_4+c$. Then $i_2+i_3+b+b^{'}+b = i_2+i_4+c$. Thus $i_2 + b + b^{'} + i_3 + b = i_2 + i_4 + c$. Hence $i_1 + a + i_5 = i_2 + i_4 + c$ for some $i_5 \in I$. Thus $i_1 + a + a^{'} + a + i_5 = i_2 + i_4 + c$. Then $i_1 + a + i_5 + a^{'} + a = i_2 + i_4 + c$. Thus $i_1 + i_6 + a = i_2 + i_4 + c$ for some $i_6 \in I$. Hence $a\rho c$.

Let $\Gamma/_{\rho} = \{[a] \,|a \in \Gamma\}$. Let us define '+' in $\Gamma/_{\rho}$ as $[a]+[b] = [a+b]$ and the map $S\times\Gamma/_{\rho} \to \Gamma/_{\rho}$ as $s\,[a] = [sa]$ for all $a, b \in \Gamma$ and $s \in S$. Suppose that $[a] = [a_1]$ and $[b] = [b_1]$ for some $a, a_1, b, b_1 \in \Gamma$. Then $i_1+a = i_2+a_1$ and $i_3+b = i_4+b_1$ for some $i_1, i_2, i_3, i_4 \in I$. Now $i_1+a+i_3+b = i_2+a_1+i_4+b_1$. Thus, $i_1+a+a^{'}+a+i_3+b = i_2+a_1+a_1^{'}+a_1+i_4+b_1$. Hence $i_1+a+i_3+a^{'}+a+b = i_2+a_1+i_4+a_1^{'}+a_1+b_1$. Then $i_1+i_5+a+b = i_2+i_6+a_1+b_1$ for some $i_5, i_6 \in I$. Thus, $[a+b] = [a_1+b_1]$.

Suppose that $[a] = [a_1]$ for some $a, a_1 \in \Gamma$. Then $i_1 + a = i_2 + a_1$ for some $i_1, i_2 \in I$. Let $s \in S$. Since $s(i_1+a)+(sa)^{'} \in I$ and $s(i_2+a_1)+(sa_1)^{'} \in I$, we have $s(i_1 + a) + (sa)^{'} + sa = i_3 + sa$ and $s(i_2 + a_1) + (sa_1)^{'} + sa_1 = i_4 + sa_1$ for some $i_3, i_4 \in I$. Let $e = (sa)^{'} + sa$ and $e_1 = (sa_1)^{'} + sa_1$. Thus, $s(i_1 + a) + e = i_3 + sa$ and $s(i_2 + a_1) + e_1 = i_4 + sa_1$. Since $i_1 + a = i_2 + a_1$, we have $a_2 + e = i_3 + sa$ and $a_2 + e_1 = i_4 + sa_1$ where $a_2 = s(i_1 + a) \in \Gamma$. Therefore, $a_2 + e + e_1 = i_5 + sa$ and $a_2 + e + e_1 = i_6 + sa_1$ for some $i_5, i_6 \in I$. Thus, $i_5 + sa = i_6 + sa_1$. Hence $[sa] = [sa_1]$. Thus, $\Gamma/_{\rho}$ is an S-semigroup.

Next we define $\Psi : \Gamma \to \Gamma/_{\rho}$ as $\Psi(\gamma) = [\gamma]$, $\gamma \in \Gamma$. Clearly Ψ is an S- homomorphism. Let K be the kernel. Take $k \in K$. Then $\Psi(k) = [0]$ implies $[k] = [0]$ implies $k\rho 0$. Hence $i_1 + k = i_2 + 0$ for some $i_1, i_2 \in I$. It follows that $i_1 + k = i_2$. Then $i_1^{'}+i_1+k = i_1^{'}+i_2$. Let $i_1^{'}+i_2 = i_3$. Hence $i_1^{'}+i_1+k = i_3$ implies $i_1^{'}+i_1+k \in I$. Since $i_1^{'} + i_1 \in E^{+}(\Gamma)$, we have $k \in I$. Therefore, $K \subseteq I$. Clearly $I \subseteq K$. Hence $K = I$. Therefore, I is the kernel of an S-homomorphism. □

4. Generalized linear sequential machine

Definition 4.1. *A semiautomaton is a triple* ***S*** $= (Q, A, F)$, *where* Q *is a state set,* A *is an input set and* $F : Q \times A \longrightarrow Q$ *is a state-transition function. If* Q *is an inverse semigroup (we always write it additively), we call* ***S*** *an inverse semigroup-semiautomaton and abbreviate this by ISA.*

For $q \in Q$ and $a \in A$ we interpret $F(q, a)$ as the new state obtained from the old state q by means of the input a. We extend A to the free monoid A^{*} over A consisting of all finite sequences of elements of A, including the empty sequence $\wedge$.

We define the function $f_a : Q \longrightarrow Q$ by

$$f_\wedge(q) = q,$$
$$f_a(q) = F(q,a) \text{ for all } a \in A$$
$$f_{xa}(q) = F(f_x(q),a) \text{ for all } x \in A^*, a \in A.$$

Note that $f_{a_1a_2} = f_{a_2}f_{a_1}$, $a_1, a_2 \in A^*$.

Now we discuss two special cases.

The homomorphism case: Let Q and A be additive inverse semigroups with 0 and $F : Q \times A \longrightarrow Q$ be a homomorphism. Now $f_a(q) = F(q,a) = F((q,0) + (0_Q,a)) = F(q,0) + F(0_Q,a) = f_0(q) + f_a(0_Q)$. Hence $f_a = f_0 + \overline{f}_a$, where f_0 is a homomorphism (i.e. a distributive element in $M(Q)$), $\overline{f}_a$ is the map with constant value $f_a(0_Q)$. Then **S** is called a homomorphic ISA.

Proposition 4.1. *For $x = a_1a_2...a_n \in A^*$,*

$$f_x = f_0^n + (f_0^{n-1}\overline{f}_{a_1} + f_0^{n-2}\overline{f}_{a_2} + ... + f_0\overline{f}_{a_{n-1}} + \overline{f}_{a_n}),$$

where $\overline{f}_a : Q \longrightarrow Q$ is the constant map with $\overline{f}_a(q) = f_a(0_Q)$ for all $q \in Q$.

Proof. We prove this result by induction on the length of the string x.

Let $a \in A$ and $q \in Q$. Now $f_a(q) = F(q,a) = F(q,0)+F(0_Q,a) = f_0(q)+f_a(0_Q)$. Then $f_a = f_0 + \overline{f}_a$, so that the result is true for $n = 1$. Assume that the result is true for $n = k-1$, i.e., $f_{a_1a_2...a_{k-1}} = f_0^{k-1}+(f_0^{k-2}\overline{f}_{a_1}+f_0^{k-3}\overline{f}_{a_2}+...+f_0\overline{f}_{a_{k-2}}+\overline{f}_{a_{k-1}})$. Now

$$\begin{aligned} f_{a_1a_2...a_k} &= f_{a_k}f_{a_1a_2...a_{k-1}} = (f_0 + \overline{f}_{a_k})f_{a_1a_2...a_{k-1}} = f_0f_{a_1a_2...a_{k-1}} + \overline{f}_{a_k}f_{a_1a_2...a_{k-1}} \\ &= f_0(f_0^{k-1} + (f_0^{k-2}\overline{f}_{a_1} + f_0^{k-3}\overline{f}_{a_2} + \ldots + f_0\overline{f}_{a_{k-2}} + \overline{f}_{a_{k-1}})) + \overline{f}_{a_k} \\ &= f_0^k + f_0^{k-1}\overline{f}_{a_1} + f_0^{k-2}\overline{f}_{a_2} + \ldots + f_0\overline{f}_{a_{k-1}} + \overline{f}_{a_k}. \end{aligned}$$

Hence the result by induction. □

The linear case: The linear case is a special case of the homomorphism case in which Q and A are R-semimodules for some semiring R and F is R-homomorphism.

Let $M = \{f_x | x \in A^*\}$. Clearly M is a submonoid of $M_{aff}(Q)$. Note that $M_d = \{f_0^n | n \geq 1\}$ is the endomorphism part of M.

Definition 4.2. *Let **S** $= (Q, A, F)$ be a ISA. The subnear-semiring $N(\boldsymbol{S})$ of $M_{aff}(Q)$ generated by M is called the syntactic near-semiring of **S**.*

Theorem 4.1. *Every non-zero element of $N(\boldsymbol{S})$ can be written as $\sum_{i=1}^{n} f_{x_i}$ for $f_{x_i} \in M$.*

Proof. Let $f = \sum_{i=1}^{n} f_{x_i}$ and $g = \sum_{j=1}^{m} f_{y_j}$ where $f_{x_i}, f_{y_j} \in M$. Clearly $N(\mathbf{S})$ is closed with respect to addition. Now

$$\begin{aligned} fg &= \left(\sum_{i=1}^{n} f_{x_i}\right)\left(\sum_{j=1}^{m} f_{y_j}\right) = \left(\sum_{i=1}^{n}(f_0^{n_i} + \overline{\overline{f}}_{x_i})\right)\left(\sum_{j=1}^{m} f_{y_j}\right) \\ &= \sum_{i=1}^{n}(f_0^{n_i}\sum_{j=1}^{m} f_{y_j} + \overline{\overline{f}}_{x_i}) = \sum_{i=1}^{n}(f_0^{n_i}\sum_{j=1}^{m-1} f_{y_j} + f_0^{n_i} f_{y_n} + \overline{\overline{f}}_{x_i}) \\ &= \sum_{i=1}^{n}(f_0^{n_i}\sum_{j=1}^{m-1} f_{y_j} + (f_0^{n_i} + \overline{\overline{f}}_{x_i})f_{y_n}) = \sum_{i=1}^{n}(f_0^{n_i}\sum_{j=1}^{m-1} f_{y_j} + f_{x_i} f_{y_n}). \end{aligned}$$

Since the above expression is a finite sum of elements of M, $N(\mathbf{S})$ is closed with respect to multiplication. Hence the result. $\square$

We extend A to the free near-semiring $A^{\#}$ over A. If $a^{\#} = w(a_1, \ldots a_n)$ is a word in $A^{\#}$ we define $f_{w(a_1,\ldots,a_n)} = w(f_{a_1}, \ldots, f_{a_n})$ and $F^{\#}(q, a^{\#}) = f_{a^{\#}}(q)$. Thus, we get an extended simultaneous sequential ISA $\mathbf{S}^{\#} = (Q, A^{\#}, F^{\#})$.

Definition 4.3. *Let $\boldsymbol{S} = (Q, A, F)$ be an ISA and $A^{\#}$ the free near-semiring on A. $q_1 \in Q$ is accessible from $q_2 \in Q$ if there is some $\alpha \in A^{\#}$ with $f_\alpha(q_2) = q_1$. $\boldsymbol{S}$ is accessible if each state q is accessible from each other state.*

$N(\mathbf{S})$ is not only a near-semiring, but it also operates on Q.

Lemma 4.1. *Q is an $N(\boldsymbol{S})$-inverse semigroup.*

Proof. Define a map $N(\mathbf{S}) \times Q \longrightarrow Q$ as for any $n = \sum_{i=1}^{n} x_i$, $x_i \in M$, $q \in Q$, $(n, q) \mapsto nq$ which satisfies the following conditions:

(1) $\left(\sum_{i=1}^{n} x_i + \sum_{j=1}^{n} y_j\right) q = \sum_{i=1}^{n} x_i(q) + \sum_{j=1}^{n} y_j(q)$, $x_i, y_j \in M$.

(2) $\left(\sum_{i=1}^{n} x_i \sum_{j=1}^{n} y_j\right) q = \sum_{i=1}^{n} x_i(\sum_{j=1}^{n} y_j(q))$, $x_i, y_j \in M$.

(3) $0q = 0_Q$.

$\square$

Proposition 4.2. *Let **S** be an ISA. **S** is accessible if and only if Q is an $S = N(\boldsymbol{S})$-inverse semigroup with $S0_Q = Q$.*

Proof. Assume that **S** is accessible. Then Q is an $N(\mathbf{S})$-inverse semigroup with $S0_Q = Q$. Conversely, suppose that $S0_Q = Q$. Let $q_1, q_2 \in Q$. Since $S0_Q = Q$, there exists $s \in S$ such that $s0_Q = q_1$. Now $s(0q_2) = q_1$. Then $(s0)q_2 = q_1$. Let $s0 = s_1 \in S$. Hence $s_1q_2 = q_1$. Therefore, **S** is accessible. □

Definition 4.4. *An automaton is a quintuple $\mathcal{A} = (Q, A, B, F, G)$, where (Q, A, F) is a semiautomaton, B is an output set and $G : Q \times A \longrightarrow B$ is an output function of $\mathcal{A}$. If Q is an inverse semigroup, $\mathcal{A}$ is called an inverse semigroup-automaton and is denoted as IA.*

$\mathcal{A}$ is called a homomorphic IA if Q, A, B are inverse semigroups and F, G are homomorphisms. $\mathcal{A}$ is called a linear IA or linear automaton or linear sequential machine if Q, A, B are R-semimodules for some semiring R and F, G are R-homomorphisms.

Since for every automaton $\mathcal{A} = (Q, A, B, F, G)$, $\mathbf{S} = (Q, A, F)$ is a semiautomaton with the same attributes, we define $N(\mathcal{A})$ as $N(\mathbf{S})$.

5. Ideal theory applied to machines

Let A^* and B^* denote the free monoids over A and B respectively. For $q \in Q$, let $s_q : A^* \longrightarrow B^*$ be defined by $s_q(\wedge) = \wedge$, $s_q(a) = G(q, a)$, $s_q(a_1a_2) = s_q(a_1)s_{F(q,a_1)}(a_2)$ and proceed inductively with

$$s_q(a_1a_2 \dots a_n) = s_q(a_1a_2 \dots a_{n-1})G(F(q, a_1 \dots a_{n-1}), a_n).$$

Definition 5.1. $s_q : A^* \longrightarrow B^*$ *is called the sequential (input-output-) function of* $\mathcal{A}$ *at* q.

Define the relation $\sim$ on Q by $q_1 \sim q_2$ if $s_{q_1} = s_{q_2}$ for all $q_1, q_2 \in Q$.

Proposition 5.1. *Let $\mathcal{A}$ be a linear IA. Then $\sim$ is a congruence relation in the $N(\mathcal{A})$-inverse semigroup Q.*

Proof. Clearly $\sim$ is reflexive and symmetric. Assume that $q_1 \sim q_2$ and $q_2 \sim q_3$. Thus, $s_{q_1} = s_{q_2}$ and $s_{q_2} = s_{q_3}$, $q_1, q_2, q_3 \in Q$. Now $s_{q_1}(\wedge) = \wedge = s_{q_3}(\wedge)$, $s_{q_1}(a) = s_{q_3}(a)$ for all $a \in A$,

$$s_{q_1}(a_1a_2) = s_{q_1}(a_1)G(F(q_1, a_1), a_2) = s_{q_3}(a_1)G(F(q_3, a_1), a_2) = s_{q_3}(a_1a_2)$$

for all $a_1, a_2 \in A$, and so on.

Hence $s_{q_1} = s_{q_3}$. Therefore, $q_1 \sim q_3$. Thus, $\sim$ is transitive.

If $q_1 \sim q_2$ then $s_{q_1} = s_{q_2}$. Let $q \in Q$. Then $s_{q_1+q}(\wedge) = \wedge = s_{q_2+q}(\wedge)$.

Let $a \in A$. Now

$$\begin{aligned} s_{q_1+q}(a) &= G(q_1+q, a) = G(q_1, a) + G(q, a') + G(0_Q, a) \\ &= G(q_2, a) + G(q, a') + G(0_Q, a) = G(q_2+q, a) = s_{q_2+q}(a). \end{aligned}$$

Let $a_1, a_2 \in A$. Now

$$\begin{aligned} s_{q_1+q}(a_1a_2) &= s_{q_1+q}(a_1)G(F(q_1+q, a_1), a_2) \\ &= s_{q_2+q}(a_1)G((F(q_1, a_1), a_2) + (F(q, a_1'), a_2') + (F(0_Q, a_1), a_2)) \\ &= s_{q_2+q}(a_1)G((F(q_2, a_1), a_2) + (F(q, a_1'), a_2') + (F(0_Q, a_1), a_2)) \\ &= s_{q_2+q}(a_1)G(F(q_2+q, a_1), a_2) = s_{q_2+q}(a_1a_2), \end{aligned}$$

and so on. Hence $s_{q_1+q} = s_{q_2+q}$. Thus, $q_1 + q \sim q_2 + q$.

Let $a \in A$ and $n = f_{a_1a_2\ldots a_k} \in N(\mathcal{A})$. Suppose that $q_1 \sim q_2$. Now,

$$\begin{aligned} s_{nq_1}(a) &= G(nq_1, a) = G(f_{a_1a_2\ldots a_k}(q_1), a) \\ &= G(F(q_1, a_1a_2\ldots a_k), a) = G(F(q_2, a_1a_2\ldots a_k), a) \\ &= G(f_{a_1a_2\ldots a_k}(q_2), a) = s_{nq_2}(a). \end{aligned}$$

Assume that $s_{nq_1}(a_1a_2\ldots a_{n-1}) = s_{nq_2}(a_1a_2\ldots a_{n-1})$. Now,

$$\begin{aligned} s_{nq_1}(a_1a_2\ldots a_n) &= s_{nq_1}(a_1a_2\ldots a_{n-1})G(F(nq_1, a_1a_2\ldots a_{n-1}), a_n) \\ &= s_{nq_2}(a_1a_2\ldots a_{n-1})G(F(f_{a_1a_2\ldots a_k}(q_1), a_1a_2\ldots a_{n-1}), a_n) \\ &= s_{nq_2}(a_1a_2\ldots a_{n-1})G(F(F(q_1, a_1a_2\ldots a_k), a_1a_2\ldots a_{n-1}), a_n) \\ &= s_{nq_2}(a_1a_2\ldots a_{n-1})G(F(F(q_2, a_1a_2\ldots a_k), a_1a_2\ldots a_{n-1}), a_n) \\ &= s_{nq_2}(a_1a_2\ldots a_{n-1})G(F(nq_2, a_1a_2\ldots a_{n-1}), a_n) \\ &= s_{nq_2}(a_1a_2\ldots a_n). \end{aligned}$$

By induction, $s_{nq_1} = s_{nq_2}$. Hence $nq_1 \sim nq_2$. □

Let $Q_0 = \{q \in Q | q \sim 0\}$. Hereafter we assume that $e + q = q + e$ for all $e \in E^+(Q)$, $q \in Q$ and $E^+(Q) \subseteq Q_0$. If Q is a group, the above conditions are trivially satisfied.

Theorem 5.1. *If $\mathcal{A}$ is a linear IA then:*

(1) $Q_0 = \{q \in Q | q \sim 0\} \trianglelefteq_{N(\mathcal{A})} Q$;
(2) $G(q, 0) = 0_B$ *for all* $q \in Q_0$.

Proof.

(1) Let $q_1, q_2 \in Q_0$. Then $q_1 \sim 0$ and $q_2 \sim 0$. Since $q_2 \sim 0$, we have $q_2' + q_2 \sim q_2'$. Thus, $q_2' \sim q_2' + q_2 \in E^+(Q) \subseteq Q_0$ implies $q_2' \sim 0$. Hence $q_1 + q_2' \sim 0$. Let $q \in Q$ and

$q_0 \in Q_0$. Since $q_0 \sim 0$ implies $q_0+q' \sim q'$. Then $q+q_0+q' \sim q+q' \in E^+(Q) \subseteq Q_0$. Hence $q+q_0+q' \sim 0$. Let $q \in Q$, $q_0 \in Q_0$ and $n \in N(\mathcal{A})$. Since $q_0 \sim 0$, $q_0+q \sim q$. Thus, $n(q_0+q) \sim nq$. Then $n(q_0+q)+(nq)' \sim nq+(nq)' \in E^+(Q) \subseteq Q_0$. Hence $n(q_0+q)+(nq)' \sim 0$. Assume that $e+q \in Q_0$ for some $e \in E^+(Q)$. Then $e+q \sim 0$ implies $e+q+q' \sim q'$. Let $q+q' = f$. Then $e+f \sim q'$. Since $e+f \in E^+(Q) \subseteq Q_0$, we have $e+f \sim 0$. Thus, $q' \sim 0$ implies $(q')' \sim 0$. Hence $q \sim 0$.

(2) Let $q \in Q_0$. Then $q \sim 0$. Now $G(q,0) = G(0,0) = 0_B$. Hence $G(q,0) = 0_B$ for all $q \in Q_0$. □

Theorem 5.2. *Let $\mathcal{A}$ be a linear IA and $g_0 : Q \to B$, $q \mapsto g_0(q) = G(q,0)$. If $(g_0 f_0^k)(q) = (g_0 f_0^k)(q_1)$ for all $k \geq 0$ then $q \sim q_1$.*

Proof. We prove this result by induction on the length of the string $a \in A^*$. If $k = 0$ then $G(q,0) = G(q_1,0)$ for all $q, q_1 \in Q$. Let $a \in A$.

Now, $s_q(a) = G(q,a) = G(q,0)+G(0_Q,a) = G(q_1,0)+G(0_Q,a) = G(q_1,a) = s_{q_1}(a)$. Assume the result is true for $k-1$, i.e. $s_q(a_1a_2\ldots a_{k-1}) = s_{q_1}(a_1a_2\ldots a_{k-1})$. Then

$$\begin{aligned}G(f_{a_1a_2\ldots a_{k-1}}(q),a_k) &= G\left((f_0^{k-1}+(f_0^{k-2}\overline{f}_{a_1}+\ldots+\overline{f}_{a_{k-1}}))(q),a_k\right)\\ &= G(f_0^{k-1}(q),0)+G\left((f_0^{k-2}\overline{f}_{a_1}+\ldots+\overline{f}_{a_{k-1}})(q),0\right)+G(0_Q,a_k)\\ &= G(f_0^{k-1}(q_1),0)+G(f_0^{k-2}\overline{f}_{a_1}+\ldots+\overline{f}_{a_{k-1}}(q_1),0)+G(0_Q,a_k)\\ &= G(f_{a_1a_2\ldots a_{k-1}}(q_1),a_k).\end{aligned}$$

Now,

$$\begin{aligned}s_q(a_1a_2\ldots a_k) &= s_q(a_1a_2\ldots a_{k-1})G(F(q,a_1a_2\ldots a_{k-1}),a_k)\\ &= s_{q_1}(a_1a_2\ldots a_{k-1})G(f_{a_1a_2\ldots a_{k-1}}(q),a_k)\\ &= s_{q_1}(a_1a_2\ldots a_{k-1})G(f_{a_1a_2\ldots a_{k-1}}(q_1),a_k)\\ &= s_{q_1}(a_1a_2\ldots a_k).\end{aligned}$$

Hence $q \sim q_1$. □

Definition 5.2. *An IA $\mathcal{A} = (Q,A,B,F,G)$ is reduced if $\sim$ is the equality. If $\mathcal{A}$ is accessible (i.e. if (Q,A,F) is accessible) and reduced then $\mathcal{A}$ is called minimal.*

Theorem 5.3. *Let $\mathcal{A}$ be a linear IA. Then $\mathcal{A}$ is reduced if and only if ${}_{N(\mathcal{A})}Q$ has no non-zero ideals P with $g_0P = \{0_B\}$.*

Proof. Assume that ${}_{N(\mathcal{A})}Q$ has no such ideals. By Theorem 5.1, Q_0 is an ideal of ${}_{N(\mathcal{A})}Q$ with $g_0Q_0 = \{0_B\}$. Then $Q_0 = \{0\}$. Hence $\mathcal{A}$ is reduced.

Conversely suppose that $\mathcal{A}$ is reduced and that $P \trianglelefteq_{N(\mathcal{A})} Q$ has $g_0P = \{0_B\}$. Then $G(p,0) = g_0(p) = 0_B$ for all $p \in P$. Since $f_0^k(p+0)+(f_0^k(0))' \in P$ for all

$p \in P$, we have $f_0^k(p) \in P$. Then $(g_0 f_0^k)(p) = 0_B$ for all $p \in P, k \geq 0$. Therefore, $(g_0 f_0^k)(p) = 0_B = (g_0 f_0^k)(0_Q)$ for all $k \geq 0$. Thus, $p \sim 0_Q$ by Theorem 5.2. Hence $p = 0_Q$. Then $P = \{0_Q\}$. □

From Proposition 4.2 and Theorem 5.3 we get

Theorem 5.4. *Let $\mathcal{A}$ be a linear IA. Then $\mathcal{A}$ is minimal if and only if $_{N(\mathcal{A})}Q$ is zero generated and does not contain non-zero ideals which are annihilated by g_0.*

Thus, in an Automata, if Q is not necessarily group but inverse semigroup, we have extended the result obtained for group Automata to check the minimality.

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